Electron-Electron Bremsstrahlung in a Hot Plasma

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Using the differential cross section for electron-electron (e-e) bremsstrahlung exact to lowest-order perturbation theory the bremsstrahlung spectra and the total rate of energy loss of a hot electron gas are calculated. The results are compared to the nonrelativistic and extreme-relativistic approximations available and to previous work. The e-e contribution to the bremsstrahlung emission of a hydrogenic plasma is found to be not negligible at temperatures above $\approx 10^8$ K, especially in the short-wavelength part of the spectrum.

1. Introduction

The observation of the X-radiation emanating from a hot Maxwellian plasma is very useful in obtaining quantitative estimates on the physical parameters such as electron temperature and density. The basic requirement for the deduction of these quantities is the knowledge of bremsstrahlung cross sections. Whereas there are no problems concerning the X-radiation arising from electron-ion (e-i) collisions, the cross section for electron-electron (e-e) bremsstrahlung was known only in nonrelativistic (quadrupole) and in extreme-relativistic approximations [1, 2]. In order to get the bremsstrahlung spectra in the intermediate energy region, one had to interpolate between the two limiting cases [3, 4]. Another difficulty is due to the fact that the cross section for e-e bremsstrahlung should be known for arbitrary directions of the electron velocities while it is usually given in a special frame of reference, e.g., the center-of-mass system or the rest system of one of the colliding particles [1, 2]. Now a covariant formula for the cross section of e-e bremsstrahlung differential with respect to photon energy and angle is available which is exact to lowest-order perturbation theory [5]. Using this expression it is possible to compute the e-e bremsstrahlung spectra of a hot plasma and to specify the range of validity and the accuracy of the approximation formulae. Additionally it makes it feasible to calculate the total energy emitted by an electron gas of temperature T. This energy can exceed the total bremsstrahlung rate generated by electron-proton (e-p) collisions in a hydrogenic plasma at very high temperatures. The results are compared to those obtained by Maxon [4] employing the interpolation method.

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2. Bremsstrahlung Spectra

The number of photons emitted per unit time, per unit volume, per unit energy interval by an electron gas of uniform number density n_e at temperature T is given by

$$P_{ee}(k,\tau) \tag{2.1}$$
$$= \tfrac{1}{2} n_e^2 c \int d^3p_1 \int d^3p_2 \, f(\boldsymbol{p}_1) f(\boldsymbol{p}_2) \frac{\sqrt{(p_1 p_2)^2 - 1}}{\varepsilon_1 \varepsilon_2} \frac{d\sigma}{dk}$$

where c is the velocity of light, ε_1 and ε_2 are the total electron energies, and $d\sigma/dk$ is the cross section for generating a photon of energy k in the collision of two electrons with the momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$; the energies ε_1, ε_2, and k are in units of the electron rest energy $m c^2$, the momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are in units of $m c$, and $(p_1 p_2) = \varepsilon_1 \varepsilon_2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2$ is the invariant product of the four-momenta $(\varepsilon_1, \boldsymbol{p}_1)$ and $(\varepsilon_2, \boldsymbol{p}_2)$.

The relativistic Maxwell-Boltzmann equilibrium distribution of electrons at temperature T is described by [6]

$$f(\boldsymbol{p}) = \frac{e^{-\varepsilon/\tau}}{4 \pi \tau K_2(1/\tau)} \tag{2.2}$$

where $\tau = k_B T/m c^2$ (k_B = Boltzmann constant) is the non-dimensional temperature and the function K_2 is the modified Bessel function of the second kind. $f(\boldsymbol{p})$ is normalized to unity,

$$\int_\infty f(\boldsymbol{p}) \, d^3p = 1 \,. \tag{2.3}$$

In (2.1) the quantity $\sqrt{(p_1 p_2)^2 - 1}/\varepsilon_1 \varepsilon_2$ originates from the invariant expression for the flux of incident particles [7, 8], and the factor $\frac{1}{2}$ prevents from the double counting of each electron.

The cross section $d\sigma/dk$ depends, apart from ε_1, ε_2, and k, only on the angle ξ between the vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Since the distribution $f(\boldsymbol{p})$ is a function of the absolute value $p = |\boldsymbol{p}| = \sqrt{\varepsilon^2 - 1}$ of the electron momentum, most of the angle integrations in (2.1) can be performed easily. Replacing the integration variable $\cos\xi$ by $\mu \equiv (p_1 p_2) = \varepsilon_1 \varepsilon_2 - p_1 p_2 \cos\xi$, one obtains

$$P_{\rm ee}(k,\tau) = \frac{c\, n_{\rm e}^2}{[2\tau K_2(1/\tau)]^2} \int \mathrm{d}\varepsilon_1 \int \mathrm{d}\varepsilon_2\, e^{-(\varepsilon_1+\varepsilon_2)/\tau} \int \mathrm{d}\mu \sqrt{\mu^2-1}\, \frac{\mathrm{d}\sigma}{\mathrm{d}k}\,. \tag{2.4}$$

The integration intervals in (2.4) follow from the kinematics of the e-e process. All the combinations of the variables ε_1, ε_2, and μ are allowed which satisfy the condition [9]

$$k \leqq \frac{\mu - 1}{\varepsilon_1 + \varepsilon_2 - \sqrt{(\varepsilon_1+\varepsilon_2)^2 - 2(\mu+1)}}\,. \tag{2.5}$$

Since the computation of $\mathrm{d}\sigma/\mathrm{d}k$ from the differential cross section $\mathrm{d}^2\sigma/\mathrm{d}k\,\mathrm{d}\Omega$ available [5] requires two angle integrations in case of arbitrary directions of the momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, one has to carry out a total of 5 integrations to obtain $P_{\rm ee}(k,\tau)$.

For "low" temperatures $\tau \ll 1$ an approximate formula for $P_{\rm ee}(k,\tau)$ can be derived. Fedyushin [10] and Garibyan [11] have calculated the cross section for the e-e quadrupole emission valid in the center-of-mass system of the colliding electrons for nonrelativistic energies $\varepsilon_1 - 1 \ll 1$, $\varepsilon_2 - 1 \ll 1$. This expression [1], generalized to arbitrary frames of reference, is given by [5]

$$\frac{\mathrm{d}\sigma_{\rm NR}}{\mathrm{d}k} = \frac{4}{15}\,\frac{\alpha r_0^2}{k}\, F\!\left(\frac{4k}{(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}\right) \tag{2.6}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant, $r_0 = e^2/m c^2$ is the classical radius of the electron, and the function F has the form

$$F(x) = \left[17 - \frac{3x^2}{(2-x)^2}\right]\sqrt{1-x} + \frac{12(2-x)^4 - 7x^2(2-x)^2 - 3x^4}{(2-x)^3}\ln\frac{1+\sqrt{1-x}}{\sqrt{x}}\,. \tag{2.7}$$

Using the ordinary Maxwell distribution

$$f_{\rm NR}(\boldsymbol{p}) = (2\pi\tau)^{-3/2}\, e^{-p^2/2\tau} \tag{2.8}$$

Equation (2.1) reduces to

$$P_{\rm ee}^{\rm NR}(k,\tau) = \tfrac{1}{2} n_{\rm e}^2 c\,(2\pi\tau)^{-3} \int \mathrm{d}^3p_1 \int \mathrm{d}^3p_2\, |\boldsymbol{p}_1 - \boldsymbol{p}_2|\, \exp\{-(p_1^2+p_2^2)/2\tau\}\, \frac{\mathrm{d}\sigma_{\rm NR}}{\mathrm{d}k}\,. \tag{2.9}$$

It is convenient to introduce the new variables [12] $\boldsymbol{p} = \boldsymbol{p}_1 - \boldsymbol{p}_2$ and $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$. Then the integration over $\boldsymbol{P}$ can be carried out, and the result is

$$P_{\rm ee}^{\rm NR}(k,\tau) = \frac{8\pi}{15}\,\alpha r_0^2 c\, n_{\rm e}^2 (\pi\tau)^{-3/2} k \int_0^1 \frac{F(x)}{x^3}\, e^{-k/(\tau x)}\,\mathrm{d}x\,. \tag{2.10}$$

This formula has been derived by Maxon and Corman [12]. The integral

$$I(k/\tau) = \int_0^1 \frac{F(x)}{x^3}\, e^{-k/(\tau x)}\,\mathrm{d}x \tag{2.11}$$

can be computed numerically. The values for $0.025 \leqq k/\tau \leqq 5$ are given in Table 1.

For temperatures up to $T = 10^8$ K the nonrelativistic formula (2.10) is a very good approximation to Equation (2.4). The values of $P_{\rm ee}^{\rm NR}(k,\tau)$ are slightly too low throughout the spectrum. At $k_{\rm B}T = 10$ keV corresponding to $T \approx 1.16 \cdot 10^8$ K the relative error is less than 1% for the photon energies above 1 keV.

The dipole spectrum emitted by a nonrelativistic hydrogenic plasma at temperature T is given by [13, 14]

$$P_{\rm ep}^{\rm NR}(k,\tau) = \frac{16}{3}\,\alpha r_0^2 c\,\frac{n_{\rm e} n_{\rm p}}{k} \cdot \sqrt{2/(\pi\tau)}\, e^{-k/2\tau} K_0(k/2\tau) \tag{2.12}$$

where $n_{\rm p}$ is the proton number density and K_0 the modified Bessel function of the second kind. At temperatures $k_{\rm B}T \gtrsim 20$ keV the contributions of higher multipoles to the bremsstrahlung spectrum become important. They are most simply included by employing the correction factor

$$R = 1/(B - v\,k) \tag{2.13}$$

derived by Quigg [14] who has given a table with the parameters B and v as a function of τ. Then

$$P_{\rm ep}(k,\tau) = R \cdot P_{\rm ep}^{\rm NR}(k,\tau)\,. \tag{2.14}$$

In the nonrelativistic approximation the ratio of e-e (quadrupole) to e-p (dipole) bremsstrahlung emis-

Table 1. Function $I(k/\tau) = \int\limits_0^1 \frac{F(x)}{x^3} e^{-k/(\tau x)} \, dx$.

k/τ	0.025	0.050	0.075	0.10	0.20	0.30	0.40	0.5	
$I(k/\tau)$	$1.311 \cdot 10^5$	$2.910 \cdot 10^4$	$1.193 \cdot 10^4$	6290	1299	497.8	245.9	139.5	
k/τ	0.6	0.7	0.8	0.9	1.0	1.2	1.4	1.6	1.8
$I(k/\tau)$	86.44	56.89	39.12	27.82	20.32	11.51	6.917	4.341	2.816
k/τ	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	
$I(k/\tau)$	1.874	1.273	0.8802	0.6172	0.4380	0.3141	0.2273	0.1658	
k/τ	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0	
$I(k/\tau)$	0.1218	0.09001	0.06690	$4.997 \cdot 10^{-2}$	$3.749 \cdot 10^{-2}$	$2.825 \cdot 10^{-2}$	$2.136 \cdot 10^{-2}$	$1.621 \cdot 10^{-2}$	

sion at a particular photon energy k can be defined [12] as (setting $n_e = n_p$)

$$V^{NR}(k/\tau) = \frac{1}{\tau} \frac{P_{ee}^{NR}(k,\tau)}{P_{ep}^{NR}(k,\tau)} \tag{2.15}$$

$$= \frac{1}{10\sqrt{2}} (k/\tau)^2 \frac{I(k/\tau)}{e^{-k/2\tau} K_0(k/2\tau)}.$$

At a given temperature τ the ratio $V^{NR}(k/\tau)$ increases considerably towards the high-energy part of the bremsstrahlung spectrum [12], that is, the contribution of e-e bremsstrahlung is greatest in the short-wavelength region. For $k_B T = 10$ keV the e-e emission exceeds 10% of the e-p emission at photon energies above 40 keV.

Figure 1 shows the e-e bremsstrahlung spectrum from an electron gas at temperature $k_B T = 100$ keV, calculated according to Eq. (2.4), compared to the results of the nonrelativistic formula (2.10) and to the e-p bremsstrahlung spectrum (2.14). In this region of temperatures the values of the nonrelativistic approximation (2.10) are considerably too low. The agreement with the exact curve is best near

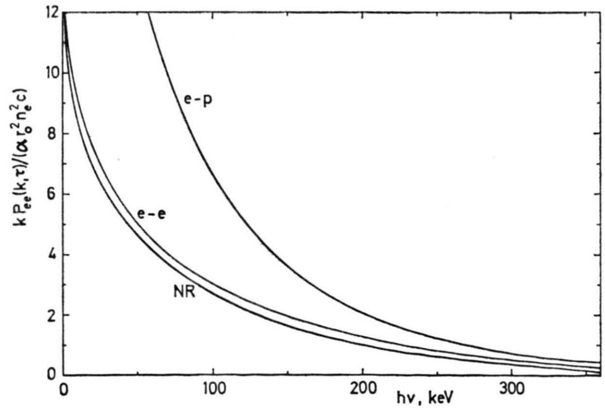

Fig. 1. Bremsstrahlung spectra for $k_B T = 100$ keV.

the photon energy $h\nu = m c^2 k = 60$ keV. The relative error made by using (2.10) increases from $\approx 8\%$ at $h\nu = 60$ keV to $\approx 30\%$ at $h\nu = 300$ keV. It is seen that the contribution of e-e bremsstrahlung to the emission of a hydrogenic plasma is no more negligible at such high temperatures, especially in the short-wavelength part of the spectrum. At $h\nu = 300$ keV, e.g., the e-e emission amounts to about 70% of the e-p emission.

The values of $P_{ee}(k,\tau)$ obtained by Maxon [4] with the aid of the interpolation method generally compare favourably to the present ones except for the points at $h\nu = 10$ keV, $h\nu = 20$ keV, and $h\nu = 200$ keV which differ by -12%, -11%, and $+16\%$, respectively. This is certainly due to his "eyeball fit" in doubly logarithmic diagrams. Therefore the interpolation method is not very reliable if one wants to have more than a rough estimate. Although Maxon's values of $P_{ep}^{NR}(k,\tau)$ sometimes differ considerably, his results for $P_{ep}(k,\tau)$ agree very well with those calculated from $P_{ep}^{NR}(k,\tau)$ by means of the correction factor (2.13).

In the preceding calculations screening effects have been ignored completely. For e-p bremsstrahlung this can be justified by computations [15] made with Debye potentials. It appears that screening by free electrons is important only for extremely low frequencies [13]. In a recent investigation on e-i bremsstrahlung in a Debye plasma Kvasnica [16] has shown that screening effects are beginning to play a part if the quantity

$$\lambda = \hbar^2/(2 m^2 c^2 D^2 \tau) \tag{2.16}$$

becomes greater than about 10^{-3}, where

$$D = \sqrt{\frac{m c^2 \tau}{4 \pi e^2 (n_e + Z^2 n_i)}} \tag{2.17}$$

is the Debye radius, Z is the atomic number, and n_i the ion density. In a hydrogenic plasma ($Z = 1$) of density $n_e = n_p = 10^{16}\,\mathrm{cm}^{-3}$, $D \approx 3.8 \times 10^{-3} \sqrt{\tau}\,\mathrm{cm}$ and $\lambda \approx 5.3 \times 10^{-17}/\tau^2$; that is, the deviations from the Coulomb-field approximation become important only below $T \approx 1400\,\mathrm{K}$, temperatures which are much lower than those considered in the present paper.

These estimates, valid for e-i bremsstrahlung, are correct in order-of-magnitude for e-e bremsstrahlung as well.

3. Bremsstrahlung Rate of Energy Loss

The total energy emitted as e-e bremsstrahlung per unit space-time volume is given by

$$W_{ee}(\tau) = m\,c^2 \int_0^\infty k\,P_{ee}(k,\tau)\,\mathrm{d}k\,. \tag{3.1}$$

Instead of integrating the photon spectrum $P_{ee}(k,\tau)$, the total energy $W_{ee}(\tau)$ can be calculated directly by using the cross section of e-e bremsstrahlung in the center-of-mass system (c. m. s.) [3, 8]. Let all quantities in the c. m. s. be labelled by the suffix cm; then the energy of a photon in an arbitrary frame of reference S is

$$k = \frac{k_{cm} + \boldsymbol{v}_{cm}\cdot\boldsymbol{k}_{cm}/c}{\sqrt{1 - v_{cm}^2/c^2}} \tag{3.2}$$

where

$$\boldsymbol{v}_{cm} = (\boldsymbol{p}_1 + \boldsymbol{p}_2)\,c/(\varepsilon_1 + \varepsilon_2) \tag{3.3}$$

is the velocity of the center-of-mass, and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, ε_1, and ε_2 are the momenta and energies, respectively, of the electrons in the system S. The total energy per unit space-time emitted in S by electrons with the momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is

$$R_{ee}(\boldsymbol{p}_1,\boldsymbol{p}_2) = n_e^2\,m\,c^3 \frac{\sqrt{(p_1 p_2)^2 - 1}}{\varepsilon_1\,\varepsilon_2} \iint k \frac{\mathrm{d}^2\sigma}{\mathrm{d}k_{cm}\,\mathrm{d}\Omega_{cm}}\,\mathrm{d}k_{cm}\,\mathrm{d}\Omega_{cm}\,. \tag{3.4}$$

Substituting (3.2) and (3.3) into (3.4), one obtains

$$R_{ee}(\boldsymbol{p}_1,\boldsymbol{p}_2) = n_e^2\,m\,c^3 \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\,\varepsilon_2} \sqrt{\frac{(p_1 p_2) - 1}{2}} \iint (k_{cm} + \boldsymbol{v}_{cm}\cdot\boldsymbol{k}_{cm}/c) \frac{\mathrm{d}^2\sigma}{\mathrm{d}k_{cm}\,\mathrm{d}\Omega_{cm}}\,\mathrm{d}k_{cm}\,\mathrm{d}\Omega_{cm}\,. \tag{3.5}$$

The differential cross section for e-e bremsstrahlung [5], $\mathrm{d}^2\sigma/\mathrm{d}k\,\mathrm{d}\Omega$, is symmetrical in the two invariant products $(k\,p_1) = \varepsilon_1 k - \boldsymbol{p}_1\cdot\boldsymbol{k}$ and $(k\,p_2) = \varepsilon_2 k - \boldsymbol{p}_2\cdot\boldsymbol{k}$. This means that in the c. m. s., where $\boldsymbol{p}_2 = -\boldsymbol{p}_1$, $\mathrm{d}^2\sigma/\mathrm{d}k\,\mathrm{d}\Omega$ is not changed by the substitution $\boldsymbol{p}_1 \to -\boldsymbol{p}_1$, i.e.,

$$\frac{\mathrm{d}^2\sigma(\boldsymbol{k}_{cm})}{\mathrm{d}k_{cm}\,\mathrm{d}\Omega_{cm}} = \frac{\mathrm{d}^2\sigma(-\boldsymbol{k}_{cm})}{\mathrm{d}k_{cm}\,\mathrm{d}\Omega_{cm}}\,. \tag{3.6}$$

Therefore the term in (3.5) containing $\boldsymbol{v}_{cm}\cdot\boldsymbol{k}_{cm}$ integrates to zero:

$$R_{ee}(\boldsymbol{p}_1,\boldsymbol{p}_2) = n_e^2\,m\,c^3 \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\,\varepsilon_2} \cdot \sqrt{\tfrac{1}{2}[(\boldsymbol{p}_1\boldsymbol{p}_2) - 1]} \int_0^{k_{max}} k_{cm} \frac{\mathrm{d}\sigma}{\mathrm{d}k_{cm}}\,\mathrm{d}k_{cm}\,. \tag{3.7}$$

The upper limit of integration, $k_{max} = p^2_{cm}/\varepsilon_{cm}$, can be expressed by the invariant product $\mu = (p_1 p_2)$:

$$k_{max} = \frac{\mu - 1}{\sqrt{2(\mu+1)}}\,. \tag{3.8}$$

The integral

$$Q_{cm} = \int_0^{k_{max}} k_{cm} \frac{\mathrm{d}\sigma}{\mathrm{d}k_{cm}}\,\mathrm{d}k_{cm} \tag{3.9}$$

is dependent on the energy ε_{cm} and the momentum p_{cm} of the electrons in the c. m. s. which, as a function of μ, have the form

$$\varepsilon_{cm} = \sqrt{\tfrac{1}{2}(\mu+1)}\,, \quad p_{cm} = \sqrt{\tfrac{1}{2}(\mu-1)}\,. \tag{3.10}$$

Finally one has to average $R_{ee}(\boldsymbol{p}_1,\boldsymbol{p}_2)$ over the Maxwell-Boltzmann distributions of the electrons to obtain

$$W_{ee}(\tau) = \tfrac{1}{2} \iint R_{ee}(\boldsymbol{p}_1,\boldsymbol{p}_2)\,f(\boldsymbol{p}_1)\,f(\boldsymbol{p}_2)\,\mathrm{d}^3p_1\,\mathrm{d}^3p_2 = n_e^2\,m\,c^3\,[2\,\tau\,K_2(1/\tau)]^{-2} \int_1^\infty \mathrm{d}\varepsilon_1 \int_1^\infty \mathrm{d}\varepsilon_2$$
$$p_1\,p_2\,(\varepsilon_1+\varepsilon_2)\,e^{-(\varepsilon_1+\varepsilon_2)/\tau} \int_{-1}^{+1} \mathrm{d}(\cos\xi)\,\sqrt{\tfrac{1}{2}(\mu-1)} \int_0^{k_{max}} k_{cm} \frac{\mathrm{d}\sigma}{\mathrm{d}k_{cm}}\,\mathrm{d}k_{cm}$$

or, introducing the new integration variable μ,

$$W_{ee}(\tau) = n_e^2\,m\,c^3 [2\,\tau\,e^{1/\tau} K_2(1/\tau)]^{-2} \int_1^\infty \mathrm{d}\varepsilon_1 \int_1^\infty \mathrm{d}\varepsilon_2\,(\varepsilon_1+\varepsilon_2)\,e^{-(\varepsilon_1+\varepsilon_2-2)/\tau} \int_{\varepsilon_1\varepsilon_2 - p_1 p_2}^{\varepsilon_1\varepsilon_2 + p_1 p_2} \mathrm{d}\mu\,\sqrt{\tfrac{1}{2}(\mu-1)}\,Q_{cm}(\mu)\,. \tag{3.11}$$

Since $Q_{cm}(\mu)$ has to be computed from the differential cross section, one needs again 5 numerical integrations to calculate $W_{ee}(\tau)$. It is, however, possible to employ an approximation procedure which requires only the 2 integrations over ε_1 and ε_2. For this purpose one has to find out an analytical expression for $Q_{cm}(\mu)$ which satisfies the condition that the integration over μ in (3.11) can be done in closed form. The procedure adopted for determining this function is to start from the nonrelativistic and the extreme-relativistic formulae for Q_{cm} and then to establish an interpolation formula comprising these two limiting cases.

In the nonrelativistic approximations the cross section (2.6) can be used resulting in (henceforth the suffix cm of the variables k, ε, and p is omitted)

$$Q_{cm}^{NR} = \frac{4}{15}\,\alpha\,r_0^2 \int_0^{p^2} F(k/p^2)\,dk = \frac{4}{15}\,\alpha\,r_0^2\,p^2 \int_0^1 F(x)\,dx = \frac{20}{3}\left(\frac{11}{3} - \frac{\pi^2}{4}\right)\alpha\,r_0^2\,p^2 \approx 8.00\,\alpha\,r_0{}^2 p^2 . \quad (3.12)$$

For very high energies the cross section has the form [5, 17]

$$\frac{d\sigma^R}{dk} = \frac{8\,\alpha\,r_0^2}{\varepsilon\,k}\left[\frac{4}{3}(\varepsilon - k) + \frac{k^2}{\varepsilon}\right] \cdot \left[\ln\left\{\frac{4\,\varepsilon^2}{k}(\varepsilon - k)\right\} - \frac{1}{2}\right], \quad (3.13)$$

resulting in

$$Q_{cm}^{R} = 16\,\alpha\,r_0^2\,\varepsilon\,[\ln(2\,\varepsilon) - \tfrac{1}{6}] . \quad (3.14)$$

An interpolation formula meeting the above conditions is given by

$$Q_{cm} \approx 8\,\alpha\,r_0^2\,\frac{p^2}{\varepsilon} \cdot \left[1 - \frac{4\,p}{3\,\varepsilon} + \frac{2}{3}\left(2 + \frac{p^2}{\varepsilon^2}\right)\ln(\varepsilon + p)\right]. \quad (3.15)$$

For $p \ll 1$ it reduces to (3.12) up to terms of order p^2, and for $\varepsilon \gg 1$ it agrees with (3.14) through second order in $1/\varepsilon$. Together the expression (3.15) yields good results at intermediate energies. This follows from a comparison with the exact values of Q_{cm} calculated by numerical integrations of the differential cross section [9]. The error made by using the approximation (3.15) is less than 1% for the kinetic electron energies $E_{cm} < 50$ keV and $E_{cm} >$ 5 MeV and is never greater than 6%.

If the quantities ε and p in (3.15) are expressed by the variable μ according to (3.10) the integral

$$J(\varepsilon_1, \varepsilon_2) = \int_{\varepsilon_1\varepsilon_2 - p_1 p_2}^{\varepsilon_1\varepsilon_2 + p_1 p_2} \sqrt{\tfrac{1}{2}(\mu - 1)}\,Q_{cm}(\mu)\,d\mu \quad (3.16)$$

can be performed analytically:

$$J(\varepsilon_1, \varepsilon_2) = 4\,\alpha\,r_0^2\left[\left(\frac{\mu}{2} - 2\right)\sqrt{\mu^2 - 1} - \frac{11}{12}\mu^2 + \frac{20}{3}\mu - \frac{8}{3}\ln(\mu + 1) + \left\{\frac{3}{2} + \left(\frac{\mu}{2} - \frac{8}{3}\,\frac{\mu + 2}{\mu + 1}\right) \cdot \sqrt{\mu^2 - 1}\right\}\ln(\mu + \sqrt{\mu^2 - 1}) + \frac{7}{4}\{\ln(\mu + \sqrt{\mu^2 - 1})\}^2\right]_{\varepsilon_1\varepsilon_2 - p_1 p_2}^{\varepsilon_1\varepsilon_2 + p_1 p_2} . \quad (3.17)$$

For $x = \mu - 1 \ll 1$ it is expedient to employ in numerical computations the form

$$J(\varepsilon_1, \varepsilon_2) \approx \alpha\,r_0^2\left[x^2\sqrt{2\,x}\left(\frac{4}{5} - \frac{x}{7}\right) + \frac{5}{18}x^4\right]_{\varepsilon_1\varepsilon_2 - p_1 p_2 - 1}^{\varepsilon_1\varepsilon_2 + p_1 p_2 - 1} \quad (3.18)$$

where the relative error compared to (3.17) is less than $5 \cdot 10^{-4}$ for $x < 0.2$.

By means of the function $J(\varepsilon_1, \varepsilon_2)$ the energy loss of a plasma due to e-e bremsstrahlung can be evaluated easily. It follows from (3.11)

$$W_{ee}(\tau) = m\,c^3\,n_e^2[2\,\tau\,e^{1/\tau}K_2(1/\tau)]^{-2}\int_1^\infty d\varepsilon_1 \int_1^\infty d\varepsilon_2\,(\varepsilon_1 + \varepsilon_2)\,e^{-(\varepsilon_1 + \varepsilon_2 - 2)/\tau}\,J(\varepsilon_1, \varepsilon_2) . \quad (3.19)$$

Utilizing the symmetry $J(\varepsilon_1, \varepsilon_2) = J(\varepsilon_2, \varepsilon_1)$ and introducing the new variables $x_1 = \varepsilon_1 - 1$ and $x_2 = \varepsilon_2 - 1$, $W_{ee}(\tau)$ can be written

$$W_{ee}(\tau) = 2\,m\,c^3\,n_e^2\,[2\,\tau\,e^{1/\tau}K_2(1/\tau)]^{-2}\int_0^\infty dx_1 \int_0^{x_1} dx_2\,(x_1 + x_2 + 2)\,e^{-(x_1 + x_2)/\tau}\,J(x_1, x_2) . \quad (3.20)$$

In the nonrelativistic limit, $\tau \ll 1$, the total energy emitted is easily derived from Equation (2.10). If the order of integrations in

$$W_{ee}^{NR}(\tau) = m\,c^2\int_0^\infty k\,P_{ee}^{NR}(k, \tau)\,dk = \frac{8\,\pi}{15}\,\alpha\,r_0^2\,m\,c^3\,n_e^2\,(\pi\,\tau)^{-3/2}\int_0^\infty dk\,k^2 \int_0^1 dx\,\frac{F(x)}{x^3}\,e^{-k/(\tau x)}$$

is changed, one obtains [3, 12]

$$W_{ee}^{NR}(\tau) = \frac{16}{15}\pi\, a\, r_0^2\, m\, c^3\, n_e^2 (\tau/\pi)^{3/2} \int_0^1 F(x)\,dx = \frac{20}{9}(44 - 3\pi^2)\, a\, r_0^2\, m\, c^3\, n_e^2\, \tau \sqrt{\frac{\tau}{\pi}} \approx 32.0\, a\, r_0^2\, m\, c^3\, n_e^2\, \tau \sqrt{\frac{\tau}{\pi}}\,. \tag{3.21}$$

For extreme-relativistic energies, $\tau \gg 1$, the formula (3.14) can be employed to get

$$J^R(\varepsilon_1, \varepsilon_2) = 8\, a\, r_0^2\, \varepsilon_1^2\, \varepsilon_2^2 [\ln(4\,\varepsilon_1\,\varepsilon_2) - \tfrac{5}{6}] \tag{3.22}$$

and

$$W_{ee}^R(\tau) = 24\, a\, r_0^2\, m\, c^3\, n_e^2\, \tau [\ln(2\,\tau) - C + \tfrac{5}{4}] \tag{3.23}$$

where $C \approx 0.5772$ is Euler's constant. The latter expression was derived by Alexanian [8].

The total energy emitted as e-p bremsstrahlung per unit space-time volume in a hydrogenic plasma is obtained from

$$W_{ep}(\tau) = n_e\, n_p\, m\, c^3 \int d^3p\, f(\mathbf{p})\, \frac{p}{\varepsilon} \int_0^{\varepsilon-1} dk\, k\, d\sigma_{ep}/dk \tag{3.24}$$

where ε and p are the energy and momentum, respectively, of the electron, and $d\sigma_{ep}/dk$ is the e-p bremsstrahlung cross section. Substituting the distribution (2.2) into (3.24) and performing the angle integrations results in

$$W_{ep}(\tau) = \frac{n_e\, n_p\, m\, c^3}{\tau\, e^{1/\tau} K_2(1/\tau)} \int_1^\infty d\varepsilon (\varepsilon^2 - 1)\, e^{-(\varepsilon-1)/\tau} \int_0^{\varepsilon-1} dk\, k \frac{d\sigma_{ep}}{dk}\,. \tag{3.25}$$

In Born approximation the integral

$$\Phi(\varepsilon) = \int_0^{\varepsilon-1} k \frac{d\sigma_{ep}}{dk}\, dk \tag{3.26}$$

can be expressed in closed form [18]. But the second integration over ε cannot be carried out analytically. Replacing $\Phi(\varepsilon)$ by an approximate function which nowhere differs by more than 0.5% from $\Phi(\varepsilon)$, Stickforth [3] has derived a formula for $W_{ep}(\tau)$,

$$W_{ep}(\tau) = \tfrac{2}{3}\, a\, r_0^2\, m\, c^3\, n_e\, n_p \frac{J_1(\tau) + J_2(\tau)}{\tau\, e^{1/\tau} K_2(1/\tau)} \tag{3.27}$$

where the functions $J_1(\tau)$ and $J_2(\tau)$ are given as series expansions.

Figure 2 shows in a doubly logarithmic scale the energy loss of a hydrogen plasma by e-e and e-p bremsstrahlung as a function of the temperature, calculated according to the formulae (3.20) and (3.27). Additionally the nonrelativistic and extreme-relativistic approximations of $W_{ee}(\tau)$ are plotted. At low temperatures $\tau \lesssim 10^{-2}$ the e-e rate is negligible compared to the e-p emission. The e-e contribution increases rapidly with temperature until the two curves cross at $\tau \approx 0.52$ or $k_B T \approx 264$ keV. In the extreme-relativistic energy region, $\tau \gg 1$, the e-p rate is given by [3, 4]

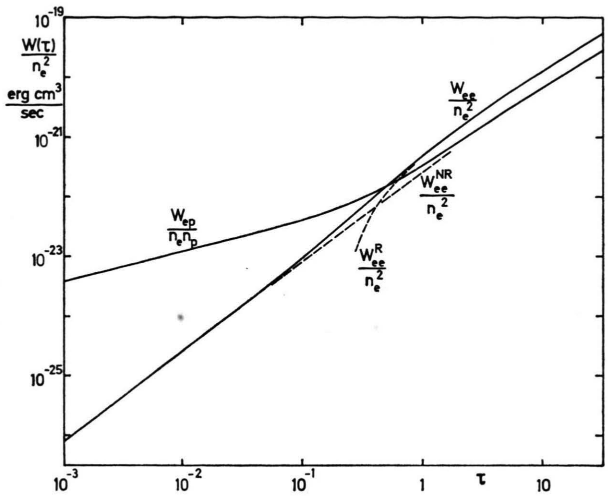

Fig. 2. Bremsstrahlung rate of energy loss in a hydrogenic plasma.

$$W_{ep}^R(\tau) = 12\, a\, r_0^2\, m\, c^3\, n_e\, n_p\, \tau [\ln(2\,\tau) + \tfrac{3}{2} - C]\,, \tag{3.28}$$

that is half of the e-e emission (3.23).

A comparison between the results of Eq. (3.20) and the nonrelativistic approximation (3.21) shows very good agreement (within 1%) for $\tau \lesssim 0.025$ or $T \lesssim 1.5 \cdot 10^8$ K. Above this temperature $W_{ee}^{NR}(\tau)$ becomes too low as can be expected from the behaviour of the spectrum $P_{ee}^{NR}(k, \tau)$. In the extreme-relativistic limit the formula (3.23) is surprisingly correct down to $\tau \approx 1$ though it was derived under the assumption $\tau \gg 1$.

It is not easy to draw the values of $W_{ee}(\tau)$ with fairly good accuracy from the doubly logarithmic diagram obtained by Maxon [4] with the aid of the interpolation method. But yet it seems that they agree quite well with the present results. On the other hand the temperature of Maxon's crossingpoint, $k_B T \approx 350$ keV, is somewhat higher than in Figure 2.

4. Conclusions

In the calculation of X-ray spectra from hot hydrogenic plasmas the contribution of e-e bremsstrahlung becomes significant above $k_B T \approx 10$ keV, especially at high photon energies. Temperatures of this order of magnitude are assumed to occur in cosmic X-ray sources such as Cyg X-1 [19] ($k_B T \approx 30$ keV). Besides it has been suggested [20] that thermal bremsstrahlung may be a source of γ-ray background flux which would require temperatures up to some MeV.

By comparison with the exact results the nonrelativistic formula for the e-e spectrum has been found to be a very good approximation up to $k_B T = 10$ keV so that it is sufficient in most practical cases. At temperatures $k_B T = 100$ keV the nonrelativistic expression becomes poorer mainly in the short-wavelength region. The total rate of energy loss by e-e bremsstrahlung in a hydrogenic plasma, $W_{ee}(\tau)$, is negligible at temperatures below $T \approx 5 \cdot 10^7$ K. At $T \approx 3 \cdot 10^9$ K the e-e emission is equal to the e-p rate and becomes twice the e-p rate at extreme-relativistic energies. The relativistic limit for $W_{ee}(\tau)$ is correct down to $\tau \approx 1$. This fact may be utilized if the e-e rate is obtained by means of interpolation.

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